VI-UG-Math(CC)-XIII (NC)

2022

Full Marks - 80

Time - 3 hours

The figures in the right-hand margin indicate marks

Answer *all* questions

Part-I

1. Answer the following : 1 × 12

a) What is the value of complex number times 'zero' ?

b) What is the domain of Arg ($\frac{1}{2}$) ?

c) What is the centre and radius of $|z-1+3i|$?

d) What is the order of a simple zero ?

e) If $z = z_0$ is a pole of order m, what is the expansion of f(z) ?

f) What is the order of zero of e^{-1/z^2} ?

g) What is the rasidue of $\frac{1}{z^3 - z^5}$ at $z = \pm 1$?

[Turn over

h) What are the solutions of $z^4 = i$.

i) What are the residue of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z = \infty$?

j) What is the value of $\int \frac{dz}{z-2}$ for $|z| = 3$?

k) If C is a unit circle about origin in positive sense what is the value of $\int_c \frac{c^{-z}}{z^2}$?

l) If C is a circle, $|z| = 1$, then what is the value of $\int \overline{z}\, dz$?

Part-II

2. Answer any ***eight*** of the following : 2 × 8

a) Reduce $(1-i)^4$ to a real number.

b) Evaluate $\int_c \frac{\sin z}{\left(z-\frac{\pi}{4}\right)^3} dz$ where c is the circle $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$?

c) Find whether $f(z) = \overline{z}$ is holomorphic ?

d) Write the function $f(z) = z^3 + z + 1$ in the form $u(x, y) + iv(x, y)$.

e) Evaluate $\int_c \frac{e^{3z}}{z+i}$ if c is the circle $|z+1+i| = 2$?

f) Evaluate $\int_c \frac{1}{(z-z_0)^2} dz$ where z_0 is any point within c ?

g) What are the zeros of $f(z) = \sin z - \cos z$.

h) Find the residue of $\frac{1}{(z^2+a^2)^2}$ at $z = ia$.

i) What are the poler of $f(z) = \tan \frac{1}{2}$.

j) Find the residue at $z = 0$ for $\frac{1}{z+z^2}$?

Part-III

3. Answer any *eight* of the following : 3 × 8

a) Show that if $z_1 z_2 z_3 = 0$ then at least one of the three factors is zero.

b) Show that the function $|z|^2$ is continuous everywhere.

c) Show that the function $e^x(\cos y + i \sin y)$ is holomorphic ?

d) Evaluate $\int_c \frac{dz}{z^2}$ where r is defined by $|z| = d$, $d > 0$?

e) Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where $c : |z-i| = 2$.

f) Evaluate $\int_c \frac{e^{ax}}{z^2+1} dz$ where c is the circle $|z| = 2$?

g) Show that the function $\frac{z^2+4}{e^z}$ has an isolated essential singularity at $z = \infty$?

h) Show that $\frac{\sin(z-1)}{z-1}$ has removable singularity at $z = 1$?

i) Calculate $\int \frac{dz}{z(z-2)^4}$ using residue formula.

j) Evaluate $\int \frac{5z-2}{z(z-1)} dz$ using Cauchy's residue formula.

Part-IV

4. a) If a function f is continuous throghout a region R that is both closed and bounded, prove that there exists a non negative real number M such that $|f(z)| \leq M$ for all points z in R, where equality holds for at least one such z. 6

OR

b) Use mathematical induction to verify "If z_1 and z_2 are any two non zero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0} \binom{n}{k} z_1^k \, z_2^{n-k}$$

$n = (1, 2 \ldots.)$ $k = (0, 1, 2 \ldots. n)$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

5. a) Prove that if f(z) is analytic in a simply connected domain D and c is any closed curve then $\int_c f(z)\,dz = 0$. 6

OR

b) Prove that if f(z) is analytic in the region D bounded by two closed curves c_1 and c_2 then

$$f(z_0) = \frac{1}{2\pi i}\left[\int_{c_1} \frac{f(z)}{z - z_0} dz - \int_{c_2} \frac{f(z)}{z - z_0} dz\right]$$

where z_0 is any point of D.

6. a) If f(z) is an analytic function in D prove that unless f(z) is identically zero, there exists a neighbourhood of each point in D throughout which the function has no zero except possibly at the point itself. 6

OR

b) State and prove Schwarz's reflection principle.

7. a) If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour c, then prove that

$$\int_c f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$ 6

OR

b) let z_0 be an isolated singular point of a function f and suppose that $f(z) = \frac{\phi z}{(z-z_0)^m}$, $m > 0$

$\phi(z)$ is analytic and non zero at z_0. Prove that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

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Part-I

1. Answer the following : 1 × 12

   a) Let G be a cyclic group of order 7. What is the order of Aut(G) ?

   b) What is the object of identity automorphism ?

   c) How many automorphisms are possible in z(+) ?

   d) Write a sub group of $z_{30} \oplus z_{12}$ isomorphic to $z_6 \oplus z_4$.

   e) Write down the elements of $U_7(105)$?

   f) What is a fixed point ?

   g) What is the equation of S_3 ?

   h) What are the conjugacy classes of D_4 ?

   i) How many sylow 5 subgroups of S_5 are there ?

   j) What is the order of 2 sylow subgroups of A_4 ?

   k) How many elements of order 10 are in $z_{100} \oplus z_{25}$?

   l) Are all sylow p-subgroups cyclic ?

Part-II

2. Answer any ***eight*** of the following : 2×8

a) Define inner outomorhism.

b) Let G be a group of order 4, G={e, a, b, ab} $a^2 = b^2 = e$, ab = ba. Determine A(G) ?

c) If G is a group on S_3. Verify whether the mapping $T : x \to x^{-1}$ is an automorphism or not.

d) Verify that a group of order 4 is isomorphic to $z_2 \oplus z_2$.

e) Write down U(105) as a product of cyclic groups ?

f) Show that $z_2 \oplus z_{30} \approx z_6 \oplus z_{10}$.

g) Define a sylow p-subgroup.

h) What are the order of all sylow p-subgroup where G has order 18 ?

i) What is the left action of G on X ?

j) If X = G, H is a subgrous of G, then show that G is an H-set under left multiplication.

Part-III

3. Answer any ***eight*** of the following 3×8

a) Construct a non abelian group of order 55.

b) Show that if $T \in A(G)$ then $T^{-1} \in A(G)$.

c) For any group G, show that the commutator subgroup G' is a characteristic subgroup.

d) Show that $z_2 \oplus z_3 \approx z_6$.

e) Determine the number of elements of order 5 in $z_{25} \oplus z_5$.

f) Verify that $z_2 \oplus z_{30} \neq z_{60}$.

g) Show that a group of order 48 is not simple.

h) Show that a group of order 33 has only one sylow 3-subgroup.

i) Let $G = GL_2(R)$ and $X = R^2$. Then show that axioms of group action are satisfied under mutliplication.

j) Show that $A(G) \approx$ cylic group of order 2.

Part-IV

4. a) Let g be a fixed element of a group G. Prove that there is a mapping $T_g : G \to G$ defined by $Tg(x) = g^{-1}xg \; \forall \; x \in G$ is an automorphism of G. 7

OR

b) Let G be a group and ϕ is an automorphism of G. If $a \in G$ is of order $O(a) > 0$ then prove that $O(\phi(a)) = O(a)$.

5. a) Prove that
$|(g_1, g_2 \ldots., g_n)| = lcm\,(|g_1|, |g_2| \ldots.|g_n|.$ 7

OR

b) Let G and H be finite cyclic group. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime .

6. a) State the prove Index theorem. 7

OR

b) Prove that if G is a finite group then G is isomorphic to some subgroup of S_n.

7. a) Let G be a finite group and P is a prime such that P divides 0(G). Then prove that G contains a subgroup of order P. 7

OR

b) Let G be a finite group and let P be dividing 0(G). Then the number of sylow P-subgroups is congruent to 1(mod P) and divides $|G|$.